Banach Space

Ex 1) Let N be a non-zero NLS (Normed Linear Space) and let $S = \{x \in N \mid \|x\| \leq 1\}$ be a L.Subspace of N. Then N is a Banach space iff S is complete

Proof: Part I

Let N be a B.S. Then we shall show that S is complete. For this we need to prove that every Cauchy Sequence in S converges in S.

Let $\langle x_n \rangle$ be a Cauchy Seq^n in S.

Then by def^n of S, we have $\rightarrow$

$$\|x_n\| \leq 1 \quad \forall n$$

Since $S \subset N$, $\therefore \langle x_n \rangle$ will be a C.S. in N. Since N is complete, $\therefore \langle x_n \rangle$ will converge in N.

So $\exists \; x \in N$ s.t. $x_n \rightarrow x$.

Now we shall show that $x \in S$.

For this we need to show that

$\|x\| \leq 1$

We have $\rightarrow$

$x_n \rightarrow x \Rightarrow \|x_n\| \rightarrow \|x\|$, $\because$ norm is a continuous function

$\therefore \|x\| = \lim\limits_{n \to \infty} \|x_n\| \leq 1$, $\because \|x_n\| \leq 1 \; \forall n$

$\therefore \|x\| \leq 1 \Rightarrow x \in S$

we have shown that C.S. $\langle x_n \rangle$ in S converges to x in S.

$\therefore$ S is complete

<u>Part III</u> Let S be complete. To show that N is a B.S.

Since N is a NLS, $\therefore$ we need to show that N is complete.

For this, we shall show that any Cauchy seqn (C.S.) in N will converge in N.

Let $\langle y_n \rangle$ be a C.S. in N.

Then by defn of C.S., we have

$$\|y_m - y_n\| \to 0 \text{ as } m, n \to \infty \to (1)$$

Define $$x_n = \frac{y_n}{\|y_n\|}, \quad \forall n$$

Then $$\|x_n\| = \frac{\|y_n\|}{\|y_n\|} = 1 \Rightarrow x_n \in S$$

Now we shall show that $\langle x_n \rangle$ is a C.S. in S. Now

$$\|x_m - x_n\| = \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$= \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_m\|} + \frac{y_n}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

[<u>Note</u>: We add & substract same term]

$$\leq \left\| \frac{y_m}{\|y_m\|} - \frac{y_n}{\|y_m\|} \right\| + \left\| \frac{y_n}{\|y_m\|} - \frac{y_n}{\|y_n\|} \right\|$$

$$, \because \|x+y\| \leq \|x\| + \|y\|$$

$$= \left\| \frac{1}{\|y_m\|} (y_m - y_n) \right\| + \left\| \left(\frac{1}{\|y_m\|} - \frac{1}{\|y_n\|} \right) y_n \right\|$$

$$= \frac{1}{\|y_m\|} \|y_m - y_n\| + \left| \frac{1}{\|y_m\|} - \frac{1}{\|y_n\|} \right| \|y_n\|$$

$$, \because \|\alpha x\| = |\alpha| \|x\|$$

$$= \frac{1}{\|y_m\|} \|y_m - y_n\| + \left| \frac{\|y_n\| - \|y_m\|}{\|y_m\| \|y_n\|} \right| \|y_n\|$$

$$= \frac{1}{\|y_m\|} \|y_m - y_n\| + \frac{\big| \|y_n\| - \|y_m\| \big|}{\|y_m\|}$$

$$\leq \frac{\|y_m - y_n\|}{\|y_m\|} + \frac{\|y_n - y_m\|}{\|y_m\|}$$

$$= \frac{2 \|y_m - y_n\|}{\|y_m\|}$$

$\to 0$ as $m, n \to \infty$, from eqn (1)

$\therefore \langle x_n \rangle$ is a C.S. in S. Since S is complete, $\therefore \exists\ x \in S$ such that

$$x_n \to x \Rightarrow \frac{y_n}{\|y_n\|} \to x \quad \to (2)$$

Since $\left| \|y_n\| - \|y_m\| \right| \leq \|y_n - y_m\|$ (4)

$\to 0$, from eqn (1)

$\therefore$ $\langle \|y_n\| \rangle$ is a C.S. of real numbers

i.e. in $\mathbb{R}$. Since $\mathbb{R}$ is complete,

$\therefore$ $\|y_n\| \to \alpha \in \mathbb{R}$.

$\therefore$ From eqn (2), $y_n \to \alpha x \in N$

Hence N is complete.

(Proved)